An exact pressure evolution equation for the incompressible Navier-Stokes equations

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In this paper the issue of the determination of the fluid pressure in incompressible fluids is addressed, with particular reference to the search of algorithms which permit to advance in time the fluid pressure without actually solving numerically the Poisson equation. Based on an inverse kinetic approach recently proposed for the incompressible Navier-Stokes equations we intend to prove that an exact evolution equation can be obtained which advances in time self-consistently the fluid pressure. The new equation is susceptible of numerical implementation in Lagrangian CFD simulation codes.

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An important aspect of computational fluid dynamics is related to the determination of the fluid pressure in isothermal incompressible fluids and particularly to the construction of an exact evolution equation for the fluid pressure which replaces the Poisson equation. This amounts to transform an elliptic type fluid equation into a suitable hyperbolic equation, a result which usually is reached only by means of an asymptotic formulation. In this paper we intend to show that an exact solution to this problem is possible when the evolution of the fluid fields is described by means of a suitable dynamical system, to be identified with the so-called Navier-Stokes (N-S) dynamical system [1]. Besides being a still unsolved mathematical problem, the issue is relevant for at least two reasons: a) the proliferation of numerical algorithms in computational fluid dynamics which reproduce the behavior of incompressible fluids only in an asymptotic sense (see below); b) the possible verification of conjectures involving the validity of appropriate equations of state for the fluid pressure. Another possible motivation is, of course, the ongoing quest for efficient numerical solution methods to be applied for the construction of the fluid fields $\{\rho, \mathbf{V}, p\}$, solutions of the initial and boundary-value problem associated to the incompressible N-S equations (INSE). For definiteness, it is convenient to recall that this is defined by the continuity, N-S and isochoricity equations

$$N\mathbf{V} = \mathbf{0}, \tag{1}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{2}$$

where the mass density ρ and the fluid pressure p are required to satisfy the inequalities

$$p(\mathbf{r},t) \geq 0, \tag{3}$$

$$\rho(\mathbf{r},t) = \rho_o > 0, \tag{4}$$

Here N is the N-S operator $N\mathbf{V} \equiv \rho \frac{D}{Dt}\mathbf{V} + \boldsymbol{\nabla} p + \mathbf{f} - \mu \nabla^2 \mathbf{V}$, with $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \boldsymbol{\nabla}$ the convective derivative, $\mathbf{f}$ denotes a suitably smooth volume force density acting on the fluid element, ρ_o is the constant mass density and $\mu > 0$ is the constant fluid viscosity. Equations (1)-(4) are assumed to admit strong solutions in an open set $\Omega \times I$, with $\Omega \subseteq \mathbb{R}^3$ the configurations space (defined as the subset of $\mathbb{R}^3$ where $\rho(\mathbf{r},t) > 0$) and $I \subset \mathbb{R}$ a possibly bounded time interval. By assumption $\{\rho, \mathbf{V}, p\}$ are continuous in the closure $\overline{\Omega}$. Hence if in $\Omega \times I$, $\mathbf{f}$ is at least $C^{(1,0)}(\Omega \times I)$, it follows necessarily that $\{\rho, \mathbf{V}, p\}$ must be at least $C^{(2,1)}(\Omega \times I)$, while the fluid pressure and velocity must satisfy respectively the Poisson and energy equations

$$\nabla^2 p = -\nabla \cdot \mathbf{f} - \rho \nabla \cdot (\mathbf{V} \cdot \boldsymbol{\nabla} \mathbf{V}), \tag{5}$$

$$\mathbf{V} \cdot N\mathbf{V} = 0. \tag{6}$$

It is well known that the choice of the Poisson solver results important in numerical simulations, since its efficient numerical solution depends critically on the number of modes or mesh points used for its discretization [2] (see also Ref.[3] and references therein indicated). In turbulent flows this number can become so large to effectively limit the size of direct numerical simulations (DNS) [4]. This phenomenon may be worsened by the algorithmic complexity of the numerical solution methods adopted for the Poisson equation. For this reason previously several alternative approaches have been devised which permit to advance in time the fluid pressure without actually solving numerically the Poisson equation. Some of these methods are *asymptotic,* i.e., to advance in time the fluid pressure they replace the exact Poisson equation with suitable algorithms or equations which hold only in an asymptotic sense (neglecting suitably small corrections), others are *exact solvers,* i.e., provide in principle rigorous solutions of INSE (and Poisson equation). The first category includes the pressure-based method (PBM) [5], the Chorin artificial compressibility

method (ACM) [6], the so-called preconditioning techniques [**?**], all based on ACM, and kinetic approaches, of which a notable example is provided by the so-called Lattice-Boltzmann (L-B) methods (for a review see for example Ref.[11] and references therein indicated). PBM is an iterative approach and one of the most widely used for incompressible flows. Its basic idea is to formulate a Poisson equation for pressure corrections, and then to update the pressure and velocity fields until the isochoricity condition (2) is satisfied in a suitable asymptotic sense. The ACM approach and the related preconditioning techniques, instead, are obtained by replacing the Poisson and N-S equations with suitable parameter-dependent evolution equations, assuming that the fluid fields depend on a fictitious pseudo-time variable τ. In dimensionless form the evolution equation for the pressure becomes in such a case $\varepsilon^2 \frac{\partial}{\partial \tau} p + \nabla \cdot \mathbf{V} = 0$, where $\varepsilon^2 > 0$ is an infinitesimal parameter. Manifestly this equation recovers only asymptotically, i.e., for $\varepsilon^2 \to 0$, the exact isochoricity condition (2). Introducing the fast variable $\overline{\tau} \equiv \tau/\varepsilon^2$, this implies that the fluid fields must be of the form $\mathbf{V}(\mathbf{r}, t, \overline{\tau}), p(\mathbf{r}, t, \overline{\tau})$ and should be assumed suitable smooth functions of $\overline{\tau}$. Therefore, for prescribed finite values of ε^2 (to be assumed suitably small), this equation permits to obtain also an *asymptotic estimate* for the fluid pressure $p(\mathbf{r}, t)$. This is expressed by the equation

$$p(\mathbf{r}, t) = \lim_{\overline{\tau} \to \infty} p(\mathbf{r}, t, \overline{\tau}) \cong \tag{7}$$

$$\cong \; p(\mathbf{r}, t, \overline{\tau} = 0) - \int_0^{\overline{\tau}^*} d\overline{\tau}' \nabla \cdot \mathbf{V}(\mathbf{r}, t, \overline{\tau}'),$$

where $\overline{\tau}^* >> 1$ is suitably defined and $p(\mathbf{r}, t, \overline{\tau} = 0)$ denotes some initial estimate for the fluid pressure. Several implementations on the Chorin algorithm are known in the literature (see for example Refs.[7, 8, 9, 10]). Customary L-B methods are asymptotic too since they recover INSE only in an approximate sense; moreover typically they rely on the introduction of an equation of state for the fluid pressure, for example, the equation of state of an ideal gas, or more refined models based on so-called non-ideal fluids [12]. This assumption, however, generally requires that the effective Mach-number characterizing the L-B approach, defined by the ratio $M^{eff} = V^{\sup}/c$ (with c denoting the discretized velocity of the test particles and $V^{\sup}$ the sup of the velocity field at time t), must result suitably small. As a consequence, in typical L-B approaches the fluid pressure can only be estimated asymptotically. However, there are other numerical approaches which in principle provide exact Poisson solvers. These include the so-called spectral methods in which the fluid fields are expanded in terms of suitable basis functions. Significant examples are the pure spectral Galerkin and Fourier methods [13] as well as the nonlinear Galerkin method [14], which are typically adopted for large-scale turbulence simulations. In these methods the construction of solution of the Poisson equation is obtained analytically. However, the series-representation of the fluid fields makes difficult the investigation of the qualitative properties of the solutions, such - for example - the search of a possible equation of state or an evolution equation for the fluid pressure.

Another approach which provides in principle an exact Poisson solver is the one recently proposed by Ellero and Tessarotto [1, 4], based on an *inverse kinetic theory* for INSE. This approach, recently applied also to quantum hydrodynamic equations [15], permits to represent the fluid fields as moments of a suitably smooth kinetic distribution function $f(\mathbf{x}, t)$ which obeys an appropriate inverse Vlasov-type kinetic equation:

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial \mathbf{x}} \cdot (\mathbf{X} f) = 0. \tag{8}$$

Here $\mathbf{X}(\mathbf{x}, t) \equiv \{\mathbf{v}, \mathbf{F}\}$ and $\mathbf{x} = (\mathbf{r}, \mathbf{v}) \in \Gamma \subseteq \overline{\Omega} \times \mathbb{R}^3$ is the state vector generated by the vector field $\mathbf{X}$, $\mathbf{v}$ is the kinetic velocity, while $\mathbf{F}(\mathbf{x},t)$ is an appropriate mean-field force obtained in Ref.[1]. In Refs. [16, 17], it has been proven that $\mathbf{F}(\mathbf{x},t)$ can be uniquely prescribed, in particular, in such a way that:

- All the fluid equations are obtained from appropriate moments of Eq.(8). As a consequence, the fluid equations as well as the initial and boundary conditions for the fluid fields are satisfied identically.
- The time evolution of the kinetic distribution function, $T_{t,t_o} f(\mathbf{x}_o) = f(\mathbf{x}(t), t)$, is determined by the classical dynamical system associated to the vector field $\mathbf{X}$, i.e.,

$$\begin{aligned} &\frac{d}{dt}\mathbf{x} = \mathbf{X}(\mathbf{x}, t) \\ &\mathbf{x}(t_o) = \mathbf{x}_o \end{aligned} \tag{9}$$

  (*N-S dynamical system*) which must hold for arbitrary initial conditions $\mathbf{x}_o = (\mathbf{r}_o, \mathbf{v}_o) \in \Gamma$.
- The solution of (9), $\mathbf{x}(t) = T_{t,t_o}\mathbf{x}_o$, which defines the N-S evolution operator T_{t,t_o}, determines uniquely a set of curves $\{\mathbf{x}(t)\} \equiv \{\mathbf{x}(t), \forall t \in I\}_{\mathbf{x}_o}$ obtained for arbitrary $(\mathbf{x}_o, t_o) \in \Gamma \times I$, which can be interpreted as *phase-space Lagrangian trajectories* associated to a set of fictitious *"test" particles*. Their projections onto the configuration space, denoted as *configuration-space Lagrangian trajectories,* are defined by the curves $\{\mathbf{r}(t)\} \equiv \{\mathbf{r}(t) \equiv T_{t,t_o}\mathbf{r}_o, \forall t \in I\}_{\mathbf{x}_o}$. By varying their initial conditions, in particular $\mathbf{r}_o \in \Omega$, the curves $\{\mathbf{r}(t)\}$ can span, by continuity, the whole set $\overline{\Omega}$.
- The fluid pressure $p(\mathbf{r}, t)$ is defined by

$$p(\mathbf{r}, t) = p_1(\mathbf{r}, t) - p_o(t), \tag{10}$$

(to be regarded as a *constitutive equation* for $p(\mathbf{r},t)$), where $p_1(\mathbf{r},t)$ is the *kinetic pressure* $p_1(\mathbf{r},t) = \int dv \frac{1}{3}u^2 f(\mathbf{x},t)$, p_o is denoted as *reduced pressure,* while $\mathbf{u}$ is the relative velocity $\mathbf{u}\equiv\mathbf{v}-\mathbf{V}(\mathbf{r},t)$.

- By definition, the reduced pressure p_o is solely a function of time, to be assumed suitably smooth and prescribed. Both $p_o(t)$ and $p_1(\mathbf{r},t)$ are strictly positive, while $p_o(t)$ in $\overline{\Omega}\times I$ is subject to the constraint $p_1(\mathbf{r},t)-p_o(t)\geq 0$.

- A particular solution of the inverse kinetic equation (8) is provided by the local Maxwellian distribution $f_M(\mathbf{x},t;\mathbf{V},p_1) = \frac{\rho_o}{(\pi)^{\frac{3}{2}}v_{th}^3}\exp\left\{-Y^2\right\}$ [where $Y^2 = \frac{\mathbf{u}^2}{vth^2}$ and $v_{th}^2 = 2p_1/\rho_o$]. In such a case, the vector field $\mathbf{F}$ reads:

$$\mathbf{F}(\mathbf{r},\mathbf{v},t) = \mathbf{a} - \frac{1}{\rho}N_0\mathbf{V} + \frac{\mathbf{u}}{2}A_0 p_1 + \frac{1}{\rho}\nabla p\left\{\frac{\mathcal{E}}{p_1}-\frac{3}{2}\right\}, \tag{11}$$

where $\mathbf{a}$ denotes the convective term $\mathbf{a}=\frac{1}{2}\mathbf{u}\cdot\nabla\mathbf{V}+\frac{1}{2}\nabla\mathbf{V}\cdot\mathbf{u}$, $\mathcal{E}$ is the relative kinetic energy density $\mathcal{E}=\rho u^2/2$, while N_0 and A_0 are the differential operators $N_0\mathbf{V} \equiv -\mathbf{f}(\mathbf{r},\mathbf{V},t)+\mu\nabla^2\mathbf{V}$ and $A_0 p_1(\mathbf{r},t) \equiv \frac{1}{p_1}\left[\frac{\partial}{\partial t}p_1+\nabla\cdot(\mathbf{V}p_1)\right]$. For an arbitrary and suitably smooth distribution function $f(\mathbf{x},t)$, the form of the vector field $\mathbf{F}$ satisfying these hypotheses has been given in Refs. [1, 16].

An interesting issue is related to the consequences of the constitutive equation (10) and of the N-S dynamical system generated by the initial value-problem (9). In this Letter we intend to prove that the fluid pressure $p(\mathbf{r},t)$ obeys an exact partial-differential equation which uniquely determines is time evolution. This is obtained by evaluating its Lagrangian derivative along an arbitrary configuration-space Lagrangian trajectory $\{\mathbf{r}(t)\}$ generated by the N-S dynamical system. The result can be stated as follows.

Assuming that the initial-boundary value problem associated to INSE admits a suitably strong solution $\{\rho,\mathbf{V},p\}$ *in the set* $\Omega\times I$*, the following statements hold:*

A) If $\mathbf{x}(t)$ *is a particular solution of Eq. (9) which holds for arbitrary* $\mathbf{r}(t)\in\Omega$ *and* $t\in I$*, along each phase-space Lagrangian trajectory* $\{\mathbf{x}(t)\}$ *defined by Eq. (9) the scalar field* $\xi(\mathbf{r},t)\equiv\mathcal{E}/p_1$ *obeys the exact evolution equation*

$$\frac{d}{dt}\xi = -\frac{1}{2}\mathbf{u}\cdot\nabla\ln p_1 \tag{12}$$

which holds for arbitrary initial conditions $\mathbf{x}_o=(\mathbf{r}_o,\mathbf{v}_o)$*, and* $\xi_o = \frac{\rho u_o^2}{2p_1(\mathbf{r}_o,t_o)}$*, with* $\mathbf{u}_o\equiv\mathbf{v}_o-\mathbf{V}(\mathbf{r}_o,t_o)$*. Here is* $\frac{d}{dt}$ *the Lagrangian derivative* $\frac{d}{dt}\equiv\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+\mathbf{F}\cdot\frac{\partial}{\partial\mathbf{v}}$*,* $\xi(\mathbf{r},t)$*, while all quantities (*$\mathbf{u}, E$ *and* p_1*) are evaluated along an arbitrary phase-space trajectory* $\{\mathbf{x}(t)\}$*.*

B) Vice versa, if the solutions $\mathbf{x}(t)=(\mathbf{r}(t),\mathbf{v}(t))$ *and* $\xi(t)$ *of Eqs.(9), (12) are known for arbitrary initial conditions* $\mathbf{x}_o=(\mathbf{r}_o,\mathbf{v}_o)$*,* $\mathbf{u}_o\equiv\mathbf{v}_o-\mathbf{V}(\mathbf{r}_o,t_o)$ *and* $\xi_o = \frac{\rho u_o^2}{2p_1(\mathbf{r}_o,t_o)}$*) and for all* $(\mathbf{r},t)\in\Omega\times I$*, it follows necessarily that in* $\Omega\times I$*,* $\{\rho,\mathbf{V},p\}$ *satisfy identically INSE.*

PROOF

Let us first prove statement A), namely that INSE and the N-S dynamical system imply necessarily the validity of Eq.(12). For this purpose we first notice that by construction Eq.(9) admits a unique solution $\mathbf{x}(t)$ for arbitrary initial conditions $\mathbf{x}_o=(\mathbf{r}_o,\mathbf{v}_o)\in\Gamma$, while the same equation can also be expressed in terms of the relative velocity $\mathbf{u}=\mathbf{v}-\mathbf{V}(\mathbf{r},t)$. This yields

$$\frac{d}{dt}\mathbf{u} = \mathbf{F} - \frac{D\mathbf{V}(\mathbf{r},t)}{Dt} - \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{V}(\mathbf{r},t) \tag{13}$$

Upon invoking the N-S equation (1) and by taking the scalar product of Eq.(13) by $\rho\mathbf{u}$, this equation implies

$$\frac{d}{dt}\mathcal{E} = \mathbf{u}\cdot\nabla p\left\{\frac{\mathcal{E}}{p_1}-\frac{1}{2}\right\} + \frac{\mathcal{E}}{p_1}\left[\frac{\partial}{\partial t}p_1+\nabla\cdot(\mathbf{V}p)\right], \tag{14}$$

which gives

$$\frac{d}{dt}\xi \equiv \frac{\partial}{\partial t}\xi + \mathbf{v}\cdot\boldsymbol{\nabla}\xi + \mathbf{F}\cdot\frac{\partial}{\partial\mathbf{v}}\xi = -\frac{1}{2p_1}\mathbf{u}\cdot\nabla p + \mathcal{E}\nabla\cdot\mathbf{V}. \tag{15}$$

As a consequence of the isochoricity condition (2) this equation reduces identically (i.e., for arbitrary initial conditions for the dynamical system) to Eq.(12). B) Vice versa, let us assume that the solutions $\mathbf{x}(t)=(\mathbf{r}(t),\mathbf{v}(t))$ and $\xi(t)$ of Eqs.(9), (12) are known for arbitrary initial conditions $\mathbf{x}_o\in\Gamma$ and $\xi_o = \frac{\rho u_o^2}{2p_1(\mathbf{r}_o,t_o)}$. In this case it follows the fluid fields necessarily must satisfy INSE in the whole set $\Omega\times I$. It suffices, in fact, to notice that by assumption the evolution operator T_{t,t_o} is known. This permits to determine uniquely the kinetic distribution function at time t, which reads [1] $f(\mathbf{x}(t),t) = f(\mathbf{x}_o,t_o)/J(\mathbf{x}(t),t)$, where $J(\mathbf{x}(t),t)$ is the Jacobian of the flow $\mathbf{x}_o\to\mathbf{x}(t)$. Hence, also its moments are uniquely prescribed, including both $\mathbf{V}(\mathbf{r},t)$ and $p(\mathbf{r},t)$, in such a way that they result at least $C^{(2,1)}(\Omega\times I)$. The inverse kinetic equation (8), thanks to the special form of $\mathbf{F}$ as given by Eq. (11) ensures that the N-S equation is satisfied identically in $\Omega\times I$ [1]. Moreover, since Eqs. (13) and (15) are by assumption fulfilled simultaneously, it follows that both the isochoricity condition (2) and the Poisson equation [Eq.(5)] must be satisfied too in $\Omega\times I$. This completes the proof.

As a basic implication, if the fluid velocity is assumed to satisfy both the N-S equation and isochoricity condition, the mass density satisfies the incompressibility condition (4), while $\{\mathbf{x}(t)\}$ is an arbitrary trajectory of

the N-S dynamical system, it follows that Eq.(12) determines uniquely the time-advancement of the fluid pressure. Hence, it provides an evolution equation for the fluid pressure, which by definition is equivalent simultaneously to the isochoricity condition and to the Poisson equation. This equation can in principle be used to determine the fluid pressure at an arbitrary position $\mathbf{r} \in \Omega$. However, since any given position can be reached by infinite phase-space (and also configuration-space) Lagrangian trajectories, it is sufficient to sample the configuration space by a suitable subset of Lagrangian trajectories (test particles), obtained by prescribing the initial condition $\mathbf{x}_o$.

The physical interpretation of the pressure evolution equation is elementary: it yields an unique prescription for the Lagrangian time derivative of the fluid pressure, which is defined in the frame which is locally co-moving with a test particle of state $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$.In particular, it is obvious that the specification of the initial kinetic velocity $\mathbf{v}_o$ remains essentially arbitrary, as well as the definition of the reduced pressure $p_o(t)$. This means that both the dimensionless ratios $M_V = V/|\mathbf{v}_o|$ and $M_p = p/p_o$, to be denoted as *velocity* and *pressure effective Mach numbers*, remain essentially arbitrary. As a consequence it is possible, in principle, to construct asymptotic solutions of Eq.(12) based on low effective-Mach numbers expansions, i.e., for which $M_V,\ M_p \ll 1$. As an illustration, let us prove that an approximate solution of this type can be obtained for p_1 (and hence for p) in the so-called *diffusive approximation,* i.e., by considering a subset of velocity space in which by assumption at time t_o the initial relative velocity $|\mathbf{u}|_{t_o}$ and the relative kinetic energy $\mathcal{E} = \rho_o u^2/2$ are assumed to satisfy the Mach-number orderings $M_V \sim O(\delta)$ and $M_p \sim O(\delta^0)$, being $\delta \ll 1$. These imply $\left.\frac{\mathcal{E}}{p_1}\right|_{t_o} \sim \frac{1}{\delta}$, $|\mathbf{u}|_{t_o} \sim \frac{1}{\delta^{1/2}}$. It follows that in an infinitesimal time interval $[t_o, t_1 = t_o + \Delta t]$, assuming $\Delta t \sim O(\delta)$, there results $\frac{d}{dt}\mathbf{u} \cong \frac{1}{\rho}\nabla p \frac{\mathcal{E}}{p_1}\left[1 + O(\delta^{1/2})\right]$ (*diffusive approximation*) which yields, by integrating it in the Euler approximation,

$$\mathbf{u}(t) - \mathbf{u}(t_o) \cong \frac{1}{\rho}\nabla p \frac{\mathcal{E}}{p_1}\Delta t\left[1 + O(\delta^{1/2})\right]. \qquad (16)$$

In the same approximation the relative kinetic energy at time t becomes

$$\mathcal{E}(t) \cong \frac{\mathcal{E}(t_o)}{1 - \mathbf{u}(t_o)\cdot\nabla \ln p_1(t_o)\Delta t}\left[1 + O(\delta^{1/2})\right]. \qquad (17)$$

As a consequence, Eq.(12) can now be used to advance in time p_1. In fact, integrating it and invoking again the Euler approximation, yields

$$\left.\frac{\mathcal{E}}{p_1}\right|_t - \left.\frac{\mathcal{E}}{p_1}\right|_{t_o} \cong -\frac{1}{2}\mathbf{u}(t_o)\cdot\nabla \ln p_1(t_o)\Delta t, \qquad (18)$$

which delivers an equation for $p_1(t)$. We stress that these features are potentially important for the construction of possible numerical algorithms based on Eq.(12). Therefore, the pressure evolution equation can in principle be adopted for the development of Lagrangian particle simulation methods in fluid dynamics. These developments will be the object of future investigations.

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